Circuits and Systems I

LECTURE #7
Bandlimited Reconstruction

Prof. Dr. Volkan Cevher
LIONS/Laboratory for Information and Inference Systems
License Info for SPFirst Slides

• This work released under a Creative Commons License with the following terms:
  • Attribution
    ▪ The licensor permits others to copy, distribute, display, and perform the work. In return, licensees must give the original authors credit.
  • Non-Commercial
    ▪ The licensor permits others to copy, distribute, display, and perform the work. In return, licensees may not use the work for commercial purposes—unless they get the licensor's permission.
  • Share Alike
    ▪ The licensor permits others to distribute derivative works only under a license identical to the one that governs the licensor's work.
• Full Text of the License
• This (hidden) page should be kept with the presentation
Outline - Today

• Today <> Section 4-4
  Section 4-5

• Next week <> BONUS EXAM REVIEW!

• Next lecture <> Section 5-1
  Section 5-2
  Section 5-3

CSI Progress Level:
Lecture Objectives

• FOLDING: a type of ALIASING
• DIGITAL-to-ANALOG CONVERSION is
  ‒ Reconstruction from samples
    • SAMPLING THEOREM applies
  ‒ Smooth Interpolation
• Mathematical Model of D-to-A
  ‒ SUM of SHIFTED PULSES
    • Linear Interpolation example
• A-to-D
  ▪ Convert $x(t)$ to **numbers** stored in memory
• D-to-A
  ▪ Convert $y[n]$ back to a “continuous-time” signal, $y(t)$
  ▪ $y[n]$ is called a “**discrete-time**” signal
Sampling $x(t)$

- **UNIFORM SAMPLING** at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$

**Shannon Sampling Theorem**
A continuous-time signal $x(t)$ with frequencies no higher than $f_{\text{max}}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\text{max}}$. 
Nyquist Rate

• “Nyquist Rate” Sampling
  – $f_s > \text{TWICE} \text{ the HIGHEST Frequency in } x(t)$
  – “Sampling above the Nyquist rate”

• BANDLIMITED SIGNALS
  – DEF: $x(t)$ has a HIGHEST FREQUENCY COMPONENT in its SPECTRUM
  – NON-BANDLIMITED EXAMPLE
    ▪ TRIANGLE WAVE is NOT BANDLIMITED
SPECTRUM for $x[n]$

- **INCLUDE ALL SPECTRUM LINES**
  - ALIASES
    - ADD INTEGER MULTIPLES of $2\pi$ and $-2\pi$
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQUENCIES

- **PLOT versus NORMALIZED FREQUENCY**
  - i.e., DIVIDE $f_o$ by $f_s$

$$\hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi l$$
Example: Spectrum

- \( x[n] = A \cos(0.2\pi n + \phi) \)
- FREQS @ 0.2\( \pi \) and -0.2\( \pi \)
- ALIASES:
  - \{2.2\( \pi \), 4.2\( \pi \), 6.2\( \pi \), ...\} & \{-1.8\( \pi \), -3.8\( \pi \), ...\}
  - EX: \( x[n] = A \cos(4.2\pi n + \phi) \)
- ALIASES of NEGATIVE FREQ:
  - \{1.8\( \pi \), 3.8\( \pi \), 5.8\( \pi \), ...\} & \{-2.2\( \pi \), -4.2\( \pi \), ...\}
\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 1 \text{ kHz} \]

\[ x[n] = A \cos(2\pi(100)(n/1000) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 1 \text{ msec} \) (1000 Hz)
\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 80 \text{kHz} \]

\[ x[n] = A \cos(2\pi(100)(n/80) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 12.5 \text{ msec} \) (80 Hz)
Folding (a type of ALIASING)

- EXAMPLE: 3 different $x(t)$; same $x[n]$

$$f_s = 1000$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

- 900 Hz “folds” to 100 Hz when $f_s = 1 \text{kHz}$
Digital Frequency $\hat{\omega}$ Again

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

Aliasing

Folded Alias

$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi \ell$$
Spectrum (Folding Case)

\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 125 \text{Hz} \]

\[ x[n] = A \cos(2\pi (100)(n/125) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 8 \text{ msec} \) (125 Hz)
**Frequency Domains**

\[ \hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi \ell \]

\[ f = \frac{\hat{\omega}}{2\pi} f_s \]
Demos from Chapter 4

• CD-ROM DEMOS
• SAMPLING DEMO (**con2dis GUI**)
  - Different Sampling Rates
    ▪ Aliasing of a Sinusoid
• STROBE DEMO
  - Synthetic vs. Real
  - Television **SAMPLES** at 30 fps in the US / 25 fps in EU
• Sampling & Reconstruction
SAMPLING GUI (con2dis)
D-to-A Reconstruction

• Create continuous $y(t)$ from $y[n]$
  - **IDEAL**
    - If you have formula for $y[n]$
    - Replace $n$ in $y[n]$ with $f_s t$
    - $y[n] = A\cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
    - $y(t) = A\cos(2\pi(800)t + \phi)$
D-to-A is AMBIGUOUS!

- ALIASING
  - Given $y[n]$, which $y(t)$ do we pick? ??
  - INFINITE NUMBER of $y(t)$
    - PASSING THRU THE SAMPLES, $y[n]$
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT

- RECONSTRUCT THE **SMOOTHEST** ONE
  - THE LOWEST FREQ, if $y[n] = \text{sinusoid}$
Spectrum (Aliasing Case)

\[ \hat{\omega} = 2\pi \frac{f}{f_s} \]

\[ f_s = 80 \text{Hz} \]

\[ x[n] = A \cos(2\pi(100)(n/80) + \varphi) \]

100-Hz Cosine Wave: Sampled with \( T_s = 12.5 \text{ msec} \) (80 Hz)
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to $x(t)$
- “CONNECT THE DOTS”
- INTERPOLATION

**INTUITIVE, conveys the idea**
Sample and Hold Device

- CONVERT $y[n]$ to $y(t)$
  - $y[k]$ should be the value of $y(t)$ at $t = kT_s$
  - Make $y(t)$ equal to $y[k]$ for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

**STAIR-STEP APPROXIMATION**
Square Pulse Case

Sampling and Zero-Order Reconstruction: $f_0 = 83 \ f_s = 200$

Original and Reconstructed Waveforms
Over-Sampling Case

Sampling and Zero-Order Reconstruction: $f_0 = 83 \, f_s = 800$

Original and Reconstructed Waveforms

EASIER TO RECONSTRUCT
Mathematical Model for D-to-A

\[ y(t) = \sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) \]

SQUARE PULSE:

\[ p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases} \]
Expand the Summation

$$\sum_{n=-\infty}^{\infty} y[n] p(t - nT_s) =$$

$$K + y[0] p(t) + y[1] p(t - T_s) + y[2] p(t - 2T_s) + K$$

- **SUM of SHIFTED PULSES** $p(t-nT_s)$
  - “**WEIGHTED**” by $y[n]$
  - CENTERED at $t=nT_s$
  - **SPACED** by $T_s$
  - RESTORES “**REAL TIME**”
Figure 4.17  Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times $T_s$. 
TRIANGULAR PULSE (2X)
Optimal Pulse?

Called “Bandlimited Interpolation”

\[ p(t) = \frac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad \text{for} \quad -\infty < t < \infty \]

\[ p(t) = 0 \quad \text{for} \quad t = \pm T_s, \pm 2T_s, \ldots \]
• Next week <> Bonus Exam Review

• LAB TIME NOW