Circuits and Systems I

LECTURE #3
The Spectrum, Periodic Signals, and the Time-Varying Spectrum

Prof. Dr. Volkan Cevher
LIONS/Laboratory for Information and Inference Systems
License Info for SPFirst Slides

- This work released under a Creative Commons License with the following terms:
  - Attribution
    - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees must give the original authors credit.
  - Non-Commercial
    - The licensor permits others to copy, distribute, display, and perform the work. In return, licensees may not use the work for commercial purposes—unless they get the licensor's permission.
  - Share Alike
    - The licensor permits others to distribute derivative works only under a license identical to the one that governs the licensor's work.
- Full Text of the License
- This (hidden) page should be kept with the presentation
Outline - Today

- Today
  - Section 3-1 – 3-3
  - Section 3-7
  - Section 3-8

- Next week
  - Section 3-4
  - Section 3-5
  - Section 3-6
  - Lab 2

**CSI Progress Level:**

- **CRAWL**
- **WALK**
- **RUN**
- **FLY**

**READ**
Lecture Objectives

- Sinusoids with **DIFFERENT** frequencies
  - SYNTHESIZE by Adding Sinusoids

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

- **SPECTRUM** Representation
  - Graphical Form shows **DIFFERENT** Freqs

`CSI` Progress Level:
Frequency is the vertical axis

Time is the horizontal axis

Figure 3.18 Sheet-music notation is a time–frequency diagram.

*Time is the horizontal axis*
Another FREQ. Diagram

- Plot Complex Amplitude vs. Freq

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]
Motivation

- Synthesize **Complicated** Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?
Fur Elise WAVEFORM

![Fur Elise Waveform Diagram]
Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) $T = 0.0065$ sec
Euler’s Formula Reversed

- Solve for cosine (or sine)

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]
\[ e^{-j\omega t} = \cos(-\omega t) + j \sin(-\omega t) \]
\[ e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t) \]
\[ e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t) \]

\[
\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})
\]
INVERSE Euler’s Formula

- Solve for cosine (or sine)

\[
\cos(\omega t) = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})
\]

\[
\sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})
\]
SPECTRUM Interpretation

• Cosine = sum of 2 complex exponentials:

\[ A \cos(7t) = \frac{A}{2} e^{j7t} + \frac{A}{2} e^{-j7t} \]

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big
SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

\[ A \sin(7t) = \frac{A}{2j} e^{j7t} - \frac{A}{2j} e^{-j7t} \]

\[ = \frac{1}{2} Ae^{-j0.5\pi} e^{j7t} + \frac{1}{2} Ae^{j0.5\pi} e^{-j7t} \]

- Positive freq. has phase = -0.5\pi
- Negative freq. has phase = +0.5\pi

\[ \frac{-1}{j} = j = e^{j0.5\pi} \]
Negative Frequency

• Is negative frequency real?
• Doppler Radar provides an example
  – Police radar measures speed by using the Doppler shift principle
  – Let’s assume 400Hz $\leftrightarrow$ 60 mph
  – $+400\text{Hz}$ means towards the radar
  – $-400\text{Hz}$ means away (opposite direction)
  – Think of a train whistle
Graphical Spectrum

EXAMPLE of SINE

$$A \sin(7t) = \frac{1}{2} Ae^{-j0.5\pi} e^{j7t} + \frac{1}{2} Ae^{j0.5\pi} e^{-j7t}$$

$$\left(\frac{1}{2} A\right) e^{j0.5\pi}$$

$$\left(\frac{1}{2} A\right) e^{-j0.5\pi}$$

AMPLITUDE, PHASE & FREQUENCY are shown
• Add the spectrum components:

\[ x(t) = 4e^{-j\pi/2} + 7e^{j\pi/3} + 10 + 7e^{-j\pi/3} + 4e^{j\pi/2} \]

What is the formula for the signal \( x(t) \)?
Gather \((A, \omega, \phi)\) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4
  - 7
  - 10

Note the conjugate phase

**DC** is another name for zero-freq component
**DC** component always has \(\phi = 0\) or \(\pi\) (for real \(x(t)\))
Add Spectrum Components-1

- **Frequencies:**
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz

- **Amplitude & Phase**
  - 4 \(-\pi/2\)
  - 7 \(+\pi/3\)
  - 10 0
  - 7 \(-\pi/3\)
  - 4 \(+\pi/2\)

\[ x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \]
Add Spectrum Components-2

\[ x(t) = 10 + \]

\[ 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \]
Simplify Components

\[ x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \]

Use Euler’s Formula to get REAL sinusoids:

\[ A\cos(\omega t + \varphi) = \frac{1}{2} Ae^{j\varphi}e^{j\omega t} + \frac{1}{2} Ae^{-j\varphi}e^{-j\omega t} \]
Final Answer

\[ x(t) = 10 + 14 \cos(2\pi(100)t - \pi/3) + 8 \cos(2\pi(250)t + \pi/2) \]

So, we get the general form:

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]
Summary: General Form

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

\[ x(t) = X_0 + \sum_{k=1}^{N} \Re\{X_k e^{j2\pi f_k t}\} \]

\[ \Re\{z\} = \frac{1}{2} z + \frac{1}{2} z^* \]

\[ x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{1}{2} X_k e^{j2\pi f_k t} + \frac{1}{2} X_k^{*} e^{-j2\pi f_k t} \right\} \]

\[ X_k = A_k e^{j\varphi_k} \]

Frequency = \( f_k \)
Example: Synthetic Vowel

- Sum of 5 Frequency Components

<table>
<thead>
<tr>
<th>$f_k$ (Hz)</th>
<th>$X_k$</th>
<th>Mag</th>
<th>Phase (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>771 + 12202j</td>
<td>12,226</td>
<td>1.508</td>
</tr>
<tr>
<td>400</td>
<td>-8865 + 28048j</td>
<td>29,416</td>
<td>1.876</td>
</tr>
<tr>
<td>500</td>
<td>48001 - 8995j</td>
<td>48,836</td>
<td>-0.185</td>
</tr>
<tr>
<td>1600</td>
<td>1657 - 13520j</td>
<td>13,621</td>
<td>-1.449</td>
</tr>
<tr>
<td>1700</td>
<td>4723 + 0j</td>
<td>4723</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 3.1:** Complex amplitudes for harmonic signal that approximates the vowel sound “ah”. 
SPECTRUM of VOWEL

- Note: Spectrum has $0.5X_k$ (except $X_{DC}$)
- Conjugates in negative frequency
SPECTRUM of VOWEL (Polar Format)

Vowel: Magnitude Spectrum

Vowel: Phase Angle Spectrum

$0.5A_k$

$\phi_k$
Figure 3.11  Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$. 
Problem Solving Skills

• **Math Formula**
  - Sum of Cosines
  - Amp, Freq, Phase

• **Recorded Signals**
  - Speech
  - Music
  - No simple formula

• **Plot & Sketches**
  - S(t) versus t
  - Spectrum

• **MATLAB**
  - Numerical
  - Computation
  - Plotting list of numbers
Lecture Objectives

- **Signals with HARMONIC Frequencies**
  - Add Sinusoids with $f_k = kf_0$

\[
x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi kf_0 t + \varphi_k)
\]

FREQUENCY can change vs. TIME
Chirps:

\[
x(t) = \cos(\alpha t^2)
\]

Introduce Spectrogram Visualization (*specgram.m*)
(*plotspec.m*)
Spectrum Diagram

- Recall Complex Amplitude vs. Freq

\[
\frac{1}{2} X_k^* = 4e^{-j\pi/2}
\]

\[
7e^{j\pi/3}
\]

\[
10
\]

\[
7e^{-j\pi/3}
\]

\[
\frac{1}{2} X_k = a_k
\]

\[
4e^{j\pi/2}
\]

\[
x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)
\]
Spectrum for Periodic Signals?

- Nearly **Periodic** in the Vowel Region
  - Period is (Approximately) \( T = 0.0065 \) sec
Periodic Signals

- Repeat every T secs
  - Definition
    \[ x(t) = x(t + T) \]
  - Example:
    \[ x(t) = \cos^2(3t) \]
    \[ T = \frac{2\pi}{3} \]
    \[ T = \frac{\pi}{3} \]
  - Speech can be “quasi-periodic”
Period of Complex Exponentials

\[ x(t) = e^{j\omega t} \]

\[ x(t + T) = x(t) \]

\[ e^{j\omega(t+T)} = e^{j\omega t} \]

\[ \Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k \]

\[ \omega = \frac{2\pi k}{T} = \left( \frac{2\pi}{T} \right) k = \omega_0 k \]

Definition: Period is T

\[ e^{j2\pi k} = 1 \]

k = integer
Harmonic Signal Spectrum

Periodic signal can only have: \( f_k = k f_0 \)

\[
x(t) = X_0 + \sum_{k=1}^{N} X_k e^{j2\pi ft} + \frac{1}{2} X^*_k e^{-j2\pi ft}
\]

\[
x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi kf_0 t + \varphi_k)
\]

\[
X_k = A_k e^{j\varphi_k}
\]

\[
f_0 = \frac{1}{T}
\]
Define Fundamental Frequency

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k) \]

\[ f_k = k f_0 \quad (\omega_0 = 2\pi f_0) \]

\[ f_0 = \text{fundamental Frequency (largest)} \]

\[ T_0 = \text{fundamental Period (shortest)} \]
What is the fundamental frequency?

10 Hz
POP QUIZ: Fundamental Freq.

Here’s another spectrum:

What is the fundamental frequency?

100 Hz?  50 Hz?
SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

IRRATIONAL SPECTRUM
Harmonic Signal (3 Freqs)

Sum of Cosine Waves with Harmonic Frequencies

Time $t$ (sec)
NON-Harmonic Signal

Sum of Cosine Waves with Nonharmonic Frequencies

NOT PERIODIC
Frequency Analysis

• **Now, a much HARDER problem**

• Given a recording of a song, have the computer write the music

  ▪ Can a machine extract frequencies?
    ▪ Yes, if we COMPUTE the spectrum for $x(t)$
      ▪ During short intervals
Time-Varying FREQUENCIES Diagram

Figure 3.18  Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis
A Simple Test Signal

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note
• SPECTROGRAM Tool
  - MATLAB function is specgram.m
  - SP-First has plotspec.m & spectgr.m

• **ANALYSIS** program
  - Takes x(t) as input &
  - Produces spectrum values $X_k$
  - Breaks x(t) into **SHORT TIME SEGMENTS**
    ▪ Then uses the FFT (Fast Fourier Transform)
R-rated: ADULTS ONLY

- **SPECTROGRAM Tool**
  - MATLAB function is `specgram.m`
  - SP-First has `plotspec.m` & `spectgr.m`

- **ANALYSIS** program
  - Takes $x(t)$ as input &
  - Produces spectrum values $X_k$
  - Breaks $x(t)$ into **SHORT TIME SEGMENTS**
    - Then uses the FFT (Fast Fourier Transform)

---

**CSI Progress Level:**

<table>
<thead>
<tr>
<th>CRAWL</th>
<th>WALK</th>
<th>RUN</th>
<th>FLY</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Crawl" /></td>
<td><img src="image2" alt="Walk" /></td>
<td><img src="image3" alt="Run" /></td>
<td><img src="image4" alt="Fly" /></td>
</tr>
</tbody>
</table>
Spectrogram Example

- Two **Constant** Frequencies: Beats

**BEAT SIGNAL:** FREQS = 672 Hz and 648 Hz

**BEATS:** Fo = 660 Hz, Fm = 12 Hz

\[ \cos(2\pi(660)t) \sin(2\pi(12)t) \]
AM Radio Signal

- Same as BEAT Notes

\[
\cos(2\pi (660)t) \sin(2\pi (12)t)
\]

\[
\frac{1}{2} \left( e^{j2\pi (660)t} + e^{-j2\pi (660)t} \right) \frac{1}{2j} \left( e^{j2\pi (12)t} - e^{-j2\pi (12)t} \right)
\]

\[
\frac{1}{4j} \left( e^{j2\pi (672)t} - e^{-j2\pi (672)t} - e^{j2\pi (648)t} + e^{-j2\pi (648)t} \right)
\]

\[
\frac{1}{2} \cos(2\pi (672)t - \frac{\pi}{2}) + \frac{1}{2} \cos(2\pi (648)t + \frac{\pi}{2})
\]
Spectrum of AM (Beat)

- 4 complex exponentials in AM:

\[ \frac{1}{4} e^{j\pi/2}, \quad \frac{1}{4} e^{-j\pi/2}, \quad \frac{1}{4} e^{j\pi/2}, \quad \frac{1}{4} e^{-j\pi/2} \]

What is the fundamental frequency?

648 Hz? 24 Hz?
Stepped Frequencies

- **C-major SCALE**: successive sinusoids
  - Frequency is constant for each note

![Frequencies of C-Major Scale](image)

- IDEAL

- Time (msec)
- Frequency (Hz)
Spectrogram of C-Scale

Sinusoids ONLY

From SPECGRAM ANALYSIS PROGRAM

ARTIFACTS at Transitions
Spectrogram of LAB SONG

Beethovens FIFTH  (Robby GRIFFIN)

Sinusoids ONLY

Analysis Frame = 40ms

ARTIFACTS at Transitions
Time-Varying Frequency

- Frequency can change \textit{vs. time}
  - Continuously, not stepped

\textbf{FREQUENCY MODULATION (FM)}

\[ x(t) = \cos(2\pi f_c t + v(t)) \]

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)
New Signal: Linear FM

- Called **Chirp** Signals (LFM)
  - Quadratic phase

\[ x(t) = A \cos(\alpha t^2 + 2\pi f_0 t + \varphi) \]

- Freq will change **LINEARLY** vs. time
  - Example of Frequency Modulation (FM)
  - Define “instantaneous frequency”
Instantaneous Frequency

- Definition

\[ x(t) = A \cos(\psi(t)) \]

\[ \Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) \]

- For Sinusoid:

\[ x(t) = A \cos(2\pi f_0 t + \varphi) \]

\[ \psi(t) = 2\pi f_0 t + \varphi \]

\[ \Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) = 2\pi f_0 \]
Instantaneous Frequency of the Chirp

- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

\[
x(t) = A \cos(\alpha t^2 + \beta t + \varphi)
\]

\[\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi\]

\[\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) = 2\alpha t + \beta\]
Chirp Spectrogram

1600 BW CHIRP CENTERED at 990 Hz
Chirp Waveform

1600 BW CHIRP CENTERED at 990 Hz

TIME (sec)
OTHER CHIRPS

• \( \psi(t) \) can be anything:

\[
x(t) = A \cos(\alpha \cos(\beta t) + \varphi)
\]

• \( \psi(t) \) could be speech or music:
  - FM radio broadcast

\[
\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) = -\alpha \beta \sin(\beta t)
\]
Sine-Wave Frequency Modulation (FM)

Look at CD-ROM Demos in Ch 3
That's all Folks!

- Next week

Section 3-4
Section 3-5
Section 3-6
Lab 2