Circuits and Systems I

LECTURE #2
Phasor Addition

Prof. Dr. Volkan Cevher
LIONS/Laboratory for Information and Inference Systems
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Outline - Today

• Today <> Section 2-6
  Lab 1!

• Next week <> Section 3-1
  Section 3-2
  Section 3-3
  Section 3-7
  Section 3-8

• Recommended self-study next week +
  – Appendix A: Complex Numbers read
  – Appendix B: MATLAB read
Lecture Objectives

- Phasors = Complex Amplitude
  - Complex Numbers represent Sinusoids

\[ z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t} \]

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - PHASOR ADDITION THEOREM
Lecture Objectives

- Phasors = Complex Amplitude
  - Complex Numbers represent Sinusoids

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - PHASOR ADDITION THEOREM

CSI Progress Level:
Do You Remember The Complex Numbers?

- To solve: $z^2 = -1$
  - $z = j$
  - Math and Physics use $z = i$
- Complex number: $z = x + jy$

Lab 2 warm-up

Cartesian coordinate system
Polar Form

• Vector Form
  - **Length** = 1
  - **Angle** = $\theta$

• Common Values
  - $j$ has angle of $0.5\pi$
  - $-1$ has angle of $\pi$
  - $-j$ has angle of $1.5\pi$
  - also, angle of $-j$ **could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is AMBIGUOUS
Polar <> Rectangular

- Relate \((x, y)\) to \((r, \theta)\)

\[
\begin{align*}
    r^2 &= x^2 + y^2 \\
    \theta &= \tan^{-1}\left(\frac{y}{x}\right)
\end{align*}
\]

- Most calculators do polar-rectangular

\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta
\end{align*}
\]

Need a notation for POLAR FORM
Euler’s Formula

• **Complex Exponential**
  
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]

\[ re^{j\theta} = r \cos(\theta) + jr \sin(\theta) \]
Complex Exponential

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

- Interpret this as a **Rotating Vector**
  - \( \theta = \omega t \)
  - Angle changes vs. time
  - ex: \( \omega = 20\pi \) rad/s
  - Rotates \( 0.2\pi \) in 0.01 secs

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]
cos = Real Part

Real Part of Euler’s
\[ \cos(\omega t) = \Re\{e^{j\omega t}\} \]

General Sinusoid
\[ x(t) = A \cos(\omega t + \varphi) \]

So,
\[ A \cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t + \varphi)}\} = \Re\{Ae^{j\varphi}e^{j\omega t}\} \]
Real Part Example

\[ A \cos(\omega t + \varphi) = \Re \left\{ Ae^{j\varphi} e^{j\omega t} \right\} \]

Evaluate: \[ x(t) = \Re \left\{ -3je^{j\omega t} \right\} \]

Answer:
\[
x(t) = \Re \left\{ (-3j)e^{j\omega t} \right\} = \Re \left\{ 3e^{-j0.5\pi} e^{j\omega t} \right\} = 3 \cos(\omega t - 0.5\pi)
\]
Complex Amplitude

General Sinusoid

\[ x(t) = A \cos(\omega t + \varphi) = \Re\{A e^{j\varphi} e^{j\omega t}\} \]

Complex AMPLITUDE = \( X \)

\[ z(t) = X e^{j\omega t} \quad X = A e^{j\varphi} \]

Then, any Sinusoid = REAL PART of \( X e^{j\omega t} \)

\[ x(t) = \Re\{X e^{j\omega t}\} = \Re\{A e^{j\varphi} e^{j\omega t}\} \]
Z DRILL (Complex Arith)

**INPUT #1**
- $r = 1$
- $\theta = 0$

**INPUT #2**
- $r = 1$
- $\theta = 0.25\pi$

**OPERATION**
- $z_1 + z_2$ (Add)

**YOUR GUESS**
- $r = 1$
- $\theta = \frac{\pi}{2}$

**Graphs**
- Complex plane with vectors representing $z_1$, $z_2$, and their sum.
How to AVOID Trigonometry

• Algebra, even complex, is **EASIER** !!!
• Can you recall $\cos(\theta_1 + \theta_2)$?
• Use: real part of $e^{j(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2)$
Recall Euler’s FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]
Real & Imaginary Part Plots

PHASE DIFFERENCE = $\pi/2$
Complex Exponential

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

- Interpret this as a Rotating Vector
  - \( \theta = \omega t \)
  - Angle changes vs. time
  - ex: \( \omega = 20\pi \) rad/s
  - Rotates \( 0.2\pi \) in 0.01 secs

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]
Rotating Phasor

See Demo on CD-ROM
Chapter 2
Real Part of Euler’s

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A \cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t+\varphi)}\}$$

$$= \Re\{Ae^{j\varphi}e^{j\omega t}\}$$
Complex Amplitude

General Sinusoid

\[ x(t) = A \cos(\omega t + \varphi) = \Re \{ Ae^{j\varphi} e^{j\omega t} \} \]

Sinusoid = REAL PART of \((Ae^{j\varphi})e^{j\omega t}\)

\[ x(t) = \Re \{ Xe^{j\omega t} \} = \Re \{ z(t) \} \]

**Complex AMPLITUDE = X**

\[ z(t) = Xe^{j\omega t} \quad X = Ae^{j\varphi} \]
POP QUIZ: Complex Amp

• Find the COMPLEX AMPLITUDE for:

\[ x(t) = \sqrt{3} \cos(77\pi t + 0.5\pi) \]

• Use EULER’s FORMULA:

\[
x(t) = \Re \left\{ \sqrt{3} e^{j(77\pi t + 0.5\pi)} \right\} \\
= \Re \left\{ \sqrt{3} e^{j0.5\pi} e^{j77\pi t} \right\} \\
X = \sqrt{3} e^{j0.5\pi}
\]
Want to Add Sinusoids

- ALL SINUSOIDS have **SAME** FREQUENCY
- HOW to GET \{Amp, Phase\} of RESULT?

\[
\begin{align*}
x_1(t) &= 1.7 \cos(2\pi(10)t + 70\pi/180) \\
x_2(t) &= 1.9 \cos(2\pi(10)t + 200\pi/180) \\
x_3(t) &= x_1(t) + x_2(t) \\
&= 1.532 \cos(2\pi(10)t + 141.79\pi/180)
\end{align*}
\]
Add Sinusoids

- Sum Sinusoid has **SAME** Frequency
Phasor Addition Rule

\[ x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A \cos(\omega_0 t + \phi) \]

Get the new complex amplitude by complex addition

\[ \sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi} \]
\[ \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re \left\{ A_k e^{j(\omega_0 t + \phi_k)} \right\} \]

\[ = \Re \left\{ \sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t} \right\} \]

\[ = \Re \left\{ \left( \sum_{k=1}^{N} A_k e^{j\phi_k} \right) e^{j\omega_0 t} \right\} \]

\[ = \Re \left\{ (A e^{j\phi}) e^{j\omega_0 t} \right\} = A \cos(\omega_0 t + \phi) \]
POP QUIZ: Add Sinusoids

- ADD THESE 2 SINUSOIDS:

\[ x_1(t) = \cos(77\pi t) \]
\[ x_2(t) = \sqrt{3} \cos(77\pi t + 0.5\pi) \]

- COMPLEX ADDITION:

\[ 1e^{j0} + \sqrt{3}e^{j0.5\pi} \]
POP QUIZ (answer)

- COMPLEX ADDITION:

\[ 1 + j\sqrt{3} = 2e^{j\pi/3} \]

- CONVERT back to cosine form:

\[ j\sqrt{3} = \sqrt{3}e^{j0.5\pi} \]

\[ x_3(t) = 2\cos(77\pi t + \frac{\pi}{3}) \]
Add Sinusoids Example

\[ x_1(t) \]

\[ x_2(t) \]

\[ x_3(t) = x_1(t) + x_2(t) \]
Convert Time-Shift to Phase

- Measure **peak times:**
  - $t_{m1}=-0.0194$, $t_{m2}=-0.0556$, $t_{m3}=-0.0394$

- Convert to **phase** ($T=0.1$)
  - $\phi_1=-\omega t_{m1} = -2\pi(t_{m1}/T) = 70\pi/180$, 
  - $\phi_2 = 200\pi/180$

- **Amplitudes**
  - $A_1=1.7$, $A_2=1.9$, $A_3=1.532$
Phasor Add: Numerical

- Convert Polar to Cartesian
  - $X_1 = 0.5814 + j1.597$
  - $X_2 = -1.785 - j0.6498$
  - sum =
  - $X_3 = -1.204 + j0.9476$

- Convert back to Polar
  - $X_3 = 1.532$ at angle $141.79\pi/180$
  - This is the sum
Add Sinusoids

\[ x_1(t) = 1.7 \cos(2\pi(10)t + 70\pi/180) \]
\[ x_2(t) = 1.9 \cos(2\pi(10)t + 200\pi/180) \]
\[ x_3(t) = x_1(t) + x_2(t) \]
\[ = 1.532 \cos(2\pi(10)t + 141.79\pi/180) \]
Next week

LAB TIME NOW!