Circuits and Systems I

LECTURE #13
Frequency Response of FIR Filters and Digital Filtering of Analog Signals

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Outline - Today

• Today <> Section 6-6
  Section 6-7
  Section 6-8

• Next week <> Final Exam Review

CSI Progress Level:

[Image of progress level icons: Crawl, Walk, Run, Fly]
LECTURE OBJECTIVES

- Two Domains: Time & Frequency
- Track the spectrum of $x[n]$ thru an FIR Filter: **Sinusoid-IN gives Sinusoid-OUT**
- **UNIFICATION**: How does frequency response affect $x(t)$ to produce $y(t)$?

![Diagram showing signal flow from $x(t)$ to $y(t)$ through an FIR filter.](image-url)
TIME & FREQUENCY

**FIR DIFFERENCE EQUATION** is the TIME-DOMAIN

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] = \sum_{k=0}^{M} h[k] x[n - k] \]

\[
H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}
\]

\[
H(e^{j\hat{\omega}}) = h[0] + h[1] e^{-j\hat{\omega}} + h[2] e^{-j2\hat{\omega}} + h[3] e^{-j3\hat{\omega}} + \cdots
\]
Ex: DELAY by 2 SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n - 2]$

\[b_k = \{0, 0, 1\}\]
\[h[n] = \delta[n - 2]\]
Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n - 2]$
GENERAL DELAY PROPERTY

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n - n_d]$.

$h[n] = \delta[n - n_d]$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k - n_d]e^{-j\hat{\omega}k} = e^{-j\hat{\omega}n_d}$$

ONLY ONE non-ZERO TERM for $k$ at $k = n_d$. 
FREQ DOMAIN --> TIME ??

• START with $H(e^{j\hat{\omega}})$ and find $h[n]$ or $b_k$

\[
H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}} \cos(\hat{\omega})
\]
FREQ DOMAIN --> TIME

\[ H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}} \cos(\hat{\omega}) \]

\[ = 7e^{-j2\hat{\omega}} (0.5e^{j\hat{\omega}} + 0.5e^{-j\hat{\omega}}) \]

\[ = (3.5e^{-j\hat{\omega}} + 3.5e^{-j3\hat{\omega}}) \]

\[ h[n] = 3.5\delta[n-1] + 3.5\delta[n-3] \]

\[ b_k = \{ 0, 3.5, 0, 3.5 \} \]
PREVIOUS LECTURE REVIEW

• **SINUSOIDAL** INPUT SIGNAL
  - OUTPUT has SAME FREQUENCY
  - DIFFERENT Amplitude and Phase

• **FREQUENCY RESPONSE** of FIR
  - MAGNITUDE vs. Frequency
  - PHASE vs. Freq
  - PLOTTING

\[ H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})} \]
PLOT of FREQ RESPONSE

Magnitude of Frequency Response of FIR Filter with Coefficients \( \{b_k\} = \{1, 2, 1\} \)

\[
H(e^{j\hat{\omega}}) = (2 + 2 \cos \hat{\omega})e^{-j\hat{\omega}}
\]

Phase Angle of Frequency Response of FIR Filter with Coefficients [1, 2, 1]
FILTER TYPES

• LOW-PASS FILTER (LPF)
  – BLURRING
  – ATTENUATES HIGH FREQUENCIES

• HIGH-PASS FILTER (HPF)
  – SHARPENING for IMAGES
  – BOOSTS THE HIGHS
  – REMOVES DC

• BAND-PASS FILTER (BPF)
LOW-PASS FILTER EXAMPLE

\[ x[n] \rightarrow \delta[n] + 2\delta[n-1] + \delta[n-2] \rightarrow y[n] \]

\[ x[n] \rightarrow 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \rightarrow y[n] \]
HIGH-PASS FILTER EXAMPLE

\[ x[n] \rightarrow \delta[n] - \delta[n-1] \rightarrow y[n] \]

\[ x[n] \rightarrow 1 - e^{-j\hat{\omega}} \rightarrow y[n] \]

Graphs showing input and output signals for the high-pass filter examples.
**BAND-PASS FILTER EXAMPLE**

\[ x[n] \rightarrow \delta[n] + 2\delta[n-1] + \delta[n-2] \rightarrow \delta[n] - \delta[n-1] \rightarrow y[n] \]

\[ x[n] \rightarrow 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \rightarrow 1 - e^{-j\hat{\omega}} \rightarrow y[n] \]

\[ x[n] \rightarrow 1 + e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}} \rightarrow y[n] \]
DIGITAL FILTERING OF ANALOG SIGNALS

- Use discrete-time filters to filter continuous-time signals that have been sampled

- What is the effect of the filter on the continuous-time input $x(t)$?

- What is the equivalent analog frequency response?
FREQUENCY SCALING

- If $|\omega| < \pi/T_s$
  - NO ALIASING
  - $|\hat{\omega}| < \pi$

$x(t) \rightarrow A\text{-to-}D \rightarrow x[n] \rightarrow H(e^{j\hat{\omega}}) \rightarrow y[n] \rightarrow D\text{-to-}A \rightarrow y(t)$

$x[n] = x(nT_s)$

$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} = \frac{2\pi f}{f_s}$
D-A FREQUENCY SCALING

- **TIME SAMPLING:**
  - RECONSTRUCT up to 0.5\(f_s\)
    - FREQUENCY SCALING
  - If input is \(x(t) = Ae^{j\phi}e^{j\omega t}\),
    - output is \(y(t) = H(e^{j(\omega T_s)})Ae^{j\phi}e^{j\omega t}\)
      for frequencies \(\omega\) such that \(-\pi/T_s < \omega < \pi/T_s\)

\[ t = nT_s \Rightarrow n \leftarrow tf_s \]

\[ \omega = \hat{\omega}f_s \]

ANALOG FREQUENCY RESPONSE
11-pt AVERAGER Example

\[
x(t) \xrightarrow{\text{A-to-D}} x[n] \xrightarrow{H(e^{j\hat{\omega}})} y[n] \xrightarrow{\text{D-to-A}} y(t)
\]

\[
y[n] = \sum_{k=0}^{10} \frac{1}{11} x[n - k]
\]

\[
H(e^{j\hat{\omega}}) = \frac{\sin\left(\frac{11}{2} \hat{\omega}\right)}{11\sin\left(\frac{1}{2} \hat{\omega}\right)} e^{-j5\hat{\omega}}
\]
11-pt AVERAGER

Magnitude of Frequency Response for 11-Point Running Averager

Phase Angle of Frequency Response for 11-Point Running Averager
11-pt AVERAGER Example

Input: 
\[ x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t - \frac{1}{2} \pi) \]

Input frequencies: 25 Hz and 250 Hz

Sampling frequency: \( f_s = 1000 \) Hz

Note: \( f_s > 2 f_{\text{max}} \) so no aliasing and \( x(t) \) can be reconstructed from \( x[n] \)
TRACK the FREQUENCIES

\[ x(t) \xrightarrow{A-to-D} x[n] \xrightarrow{H(e^{j\hat{\omega}})} y[n] \xrightarrow{D-to-A} y(t) \]

- 250 Hz  
- 0.5\pi 
- \( H(e^{j0.5\pi}) \) 
- 0.5\pi 
- 250 Hz

- 25 Hz  
- 0.05\pi 
- \( H(e^{j0.05\pi}) \) 
- 0.05\pi 
- 25 Hz

\( f_s = 1000 \text{ Hz} \)

**WARNING:** When there is aliasing, \( y(t) \) will have different frequency components than \( x(t) \)

\[
x_1(t) = \cos(2\pi(25)t) \quad , \quad T_s = \frac{1}{1000}
\]

\[
x_1(nT_s) = \cos\left(\frac{2\pi(25)n}{1000}\right) = \cos\left(\frac{\pi}{20}n\right)
\]
FREQUENCY RESPONSE OF 11-pt AVERAGER

Magnitude of Frequency Response for 11-Point Running Averager

Phase Angle of Frequency Response for 11-Point Running Averager
Input:

\[ x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t - \frac{1}{2}\pi) \]

Output:

\[ y(t) = 0.8811\cos\left(2\pi(25)t - \frac{\pi}{4}\right) + 0.0909\cos(2\pi(250)t - \pi) \]

Evaluate Output

\[ H(e^{j0.05\pi}) = 0.8811e^{-j0.25\pi} \]

\[ H(e^{j0.5\pi}) = 0.0909e^{-j0.5\pi} \]
Magnitude of Frequency Response for 11-Point Running Averager

Discrete-Time Frequency \( \hat{\omega} \)

Equivalent Continuous-Time Frequency Response for \( f_s = 1000 \)

EFFECTIVE RESPONSE

LOW-PASS FILTER

DIGITAL FILTER

\( H(e^{j\hat{\omega}}) \)
That's all Folks!

- Next week

<> Final Exam Review