Circuits and Systems I

LECTURE #11
Linearity, Time Invariance, and Convolution

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Outline - Today

- Today <> Section 5-4
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- Next week <> Section 6-1
  Section 6-2
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CSI Progress Level:
Lecture Objectives

- **GENERAL** PROPERTIES of FILTERS
  - LINEARITY
  - TIME-INVARIANCE
  - ==> CONVOLUTION

- BLOCK DIAGRAM REPRESENTATION
  - Components for Hardware
  - Connect Simple Filters Together to Build More Complicated Systems
Overview

- **IMPULSE RESPONSE**, \( h[n] \)
  - FIR case: same as \( \{b_k\} \)

- **CONVOLUTION**
  - GENERAL: \( y[n] = h[n] * x[n] \)

- GENERAL CLASS of SYSTEMS

- LINEAR and TIME-ININVARIANT

- **ALL LTI** systems have \( h[n] \) & use convolution
• CONCENTRATE on the FILTER (DSP)
• DISCRETE-TIME SIGNALS
  – FUNCTIONS of $n$, the “time index”
  – INPUT $x[n]$
  – OUTPUT $y[n]$
• BUILD UP COMPLICATED FILTERS
  – FROM SIMPLE MODULES
  – Ex: FILTER MODULE MIGHT BE 3-pt FIR
General FIR Filter

- FILTER COEFFICIENTS \(\{b_k\}\)
  - DEFINE THE FILTER

  - For example,

\[
y[n] = \sum_{k=0}^{M} b_k x[n - k]
\]

\[
b_k = \{3, -1, 2, 1\}
\]

\[
y[n] = \sum_{k=0}^{3} b_k x[n - k]
\]

\[
= 3x[n] - x[n - 1] + 2x[n - 2] + x[n - 3]
\]
MATLAB for FIR Filter

- \( yy = \text{conv}(bb, xx) \)
  - VECTOR \( bb \) contains Filter Coefficients
  - DSP-First: \( yy = \text{firfilt}(bb, xx) \)

- FILTER COEFFICIENTS \( \{b_k\} \)

\[
y[n] = \sum_{k=0}^{M} b_k x[n - k]
\]
Special Input Signals

- $x[n] = \text{SINUSOID}$
- $x[n]$ has only one \text{NON-ZERO VALUE}

\[
\delta[n] = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases}
\]
FIR Impulse Response

- Convolution = Filter Definition
  - Filter Coeffs = Impulse Response

\[
h[n] = \sum_{k=0}^{M} b_k \delta[n - k]
\]
Mathematical Formula for $h[n]$

- Use **SHIFTED** IMPULSES to write $h[n]$

$$h[n] = \delta[n] - \delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$

$$b_k = \{1, -1, 2, -1, 1\}$$
LTI: Convolution Sum

- **Output = Convolution of $x[n]$ & $h[n]$**
  - **NOTATION:**
  - Here is the FIR case:

\[ y[n] = h[n] \ast x[n] \]

\[ y[n] = \sum_{k=0}^{M} h[k]x[n - k] \]

Same as $b_k$
CONVOLUTION Example

\[ h[n] = \delta[n] - \delta[n - 1] + 2\delta[n - 2] - \delta[n - 3] + \delta[n - 4] \]
\[ x[n] = u[n] \]

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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
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<tr>
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<td>1</td>
<td>-1</td>
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<td>-1</td>
<td>1</td>
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<td>1</td>
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<td>1</td>
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<tr>
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<tr>
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<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( h[3]x[n - 3] )</td>
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<td>0</td>
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<td>-1</td>
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<td>-1</td>
</tr>
<tr>
<td>( h[4]x[n - 4] )</td>
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<td>0</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>( y[n] )</td>
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<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>...</td>
</tr>
</tbody>
</table>
GENERAL FIR FILTER

• SLIDE a Length-L WINDOW over $x[n]$

$$y[n] = \sum_{k=0}^{M} b_k x[n - k]$$

$M$-th Order FIR Filter Operation (Causal)
DCONVDEMO: MATLAB GUI

Signal
- Flipped Signal

Input

Impulse Response

Multiplication

Linear Convolution

Signal Axis:
- $o = x[k]$
- $o = h[n-k]$

Multiplication Axis:
- $x[k]h[n-k]$

Convolution Axis:
- $y[n] = \sum x[k]h[n-k]$

Get $x[n]$

Get $h[n]$

Flip $x[n]$

Flip $h[n]$

Close

Help
Pop Quiz

• FIR Filter is “FIRST DIFFERENCE”
  \[ y[n] = x[n] - x[n-1] \]
• INPUT is “UNIT STEP”

\[ u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \]

Find \( y[n] \)

\[ y[n] = u[n] - u[n - 1] = \delta[n] \]
Hardware Structures

- **INTERNAL STRUCTURE** of “FILTER”
  - WHAT COMPONENTS ARE NEEDED?
  - HOW DO WE “HOOK” THEM TOGETHER?

- **SIGNAL FLOW GRAPH NOTATION**
Hardware Atoms

• Add, Multiply & Store

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]

\[ y[n] = \beta x[n] \]

\[ y[n] = x_1[n] + x_2[n] \]

\[ y[n] = x[n - 1] \]
FIR Structure

- Direct Form

\[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]

**Figure 5.13** Block-diagram structure for the \( M \)th order FIR filter.
Moore’s Law for TI DSPs

Device Density (Log Scale)

Computation Speed

Double every 18 months?
System Properties

- **MATHEMATICAL DESCRIPTION**
- **TIME-INVARIANCE**
- **LINEARITY**
- **CAUSALITY**
  - “No output prior to input”
Time-Invariance

• IDEA:
  – “Time-Shifting the input will cause the same time-shift in the output”

• EQUIVALENTLY,
  – We can prove that
    ▪ The time origin (n=0) is picked arbitrary
TESTING Time-Invariance

Figure 5.16 Testing time-invariance property by checking the interchange of operations.
Linear System

• LINEARITY = Two Properties

• SCALING
  – “Doubling x[n] will double y[n]”

• SUPERPOSITION:
  – “Adding two inputs gives an output that is the sum of the individual outputs”
Figure 5.17  Testing linearity by checking the interchange of operations.
LTI SYSTEMS

• LTI: Linear & Time-Invariant

• COMPLETELY CHARACTERIZED by:
  – **IMPULSE RESPONSE** $h[n]$
  – **CONVOLUTION**: $y[n] = x[n] * h[n]$
    - The “rule” defining the system can ALWAYS be re-written as convolution

• FIR Example: $h[n]$ is same as $b_k$
Pop Quiz

- FIR Filter is “FIRST DIFFERENCE”
  \[ y[n] = x[n] - x[n - 1] \]

- Write output as a convolution
  - Need impulse response
    \[ h[n] = \delta[n] - \delta[n - 1] \]
  - Then, another way to compute the output:
    \[ y[n] = (\delta[n] - \delta[n - 1]) \ast x[n] \]
Cascade Systems

- Does the order of $S_1$ & $S_2$ matter?
  - NO, **LTI SYSTEMS can be rearranged !!!**
  - WHAT ARE THE FILTER COEFFS?  \{b_k\}

![Diagram of Cascade Systems](image)

**Figure 5.19**  A Cascade of Two LTI Systems.
Cascade Equivalent

- Find “overall” $h[n]$ for a cascade?

**Figure 5.19** A Cascade of Two LTI Systems.

**Figure 5.20** Switching the order of cascaded LTI systems.
Next week

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Section 6-3
Section 6-4